Particle Acceleration in Rotating Modified Hayward and Bardeen Black Holes

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Abstract

In this paper we consider rotating modified Hayward and rotating modified Bardeen black holes as particle accelerators. We investigate the center of mass energy of two colliding neutral particles with same rest masses falling from rest at infinity to near the horizons of the mentioned black holes. We investigate the range of the particle's angular momentum and the orbit of the particle. We also investigate the center of mass energy for extremal black hole.

Keywords: Black hole; Particle acceleration.

1 Introduction

Collision of particles around black holes is very interesting topic in current astrophysical research. Recently, Bañados, Silk and West (BSW) [1] have investigated the collision of two particles falling from rest at infinity into the Kerr black hole, which is known as BSW mechanism. They determined the center of mass (CM) energy in the equatorial plane, which may be high in the limiting case of extremal black hole. Further Lake [2, 3] demonstrated that the CM energy of two colliding particles diverges at the inner horizon of non-extremal Kerr black hole. A general review of particle accelerator of black hole is discussed by Harada et al [4]. Wei et al [5] also investigated that the collision of two uncharged particles around Kerr-Newmann black hole, which properly depends on the spin and charge of the black hole. Liu et al [6] demonstrated that the collision of two particles around Kerr-Taub-NUT black hole. Subsequently, Zakria and Jamil [7] investigated the CM energy of the collision for two neutral particles with different rest masses falling freely from rest at infinity in the background of a Kerr-Newman-Taub-NUT black hole. Till now, several authors [8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20] have studied the collision of particles near black holes and CM energy for the colliding particles.

It was argued that the CM energy of the colliding particles for the naked singularity diverges [21, 22, 23, 24]. Due to gravitational collapse, any astrophysical object produces either space-time

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singularity or naked singularity. Since we know that any classical black hole has a singularity. To avoid the singularity, Bardeen [25] proposed the concept of regular black hole, dubbed as Bardeen black hole and subsequently, another type of regular black hole (Hayward black hole) found [26]. Another kind of regular black hole is Ayon-Beato- Garcya (ABG) black hole [27]. Geodesic study of regular Hayward black hole has been discussed by Abhas et al [28]. The implication of rotating Hayward black hole is discussed in ref.[29]. Modified Hayward black hole metric has been proposed by Lorenzo et al [30]. Recently, Amir et al [31] studied the collision of two particles with equal masses moving in the equatorial plane near horizon of the rotating Hayward's regular black hole (as particle accelerator). Also P. Pradhan [32] studied the regular Hayward and Bardeen black holes as particle accelerator. The CM energy of the collision for charged particles in a Bardeen black hole was studied in [33]. CM energy and horizon structure for rotating Bardeen black hole are also studied in ref [34]. We now extend the above works into rotating modified Hayward and rotating modified Bardeen black holes. The CM energy and the particles orbit are investigated for two colliding neutral particles of same rest masses falling from infinity into the above mentioned black holes. Next we discuss the extremal limits of the black holes. Finally we conclude the results for particle acceleration near the black holes.

2 Rotating Black Hole Background

The rotating black hole metric can be written as

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi, \tag{1}$$

where,

$$\begin{aligned}
g_{tt} &= -\tilde{h}\tilde{f}, \\
g_{rr} &= \frac{\Sigma}{\tilde{f}\Sigma + a^2\sin^2\theta}, \\
g_{\theta\theta} &= \Sigma, \\
g_{\phi\phi} &= \sin^2\theta\left[\Sigma + a^2(2-\tilde{f})\sin^2\theta\right], \\
g_{t\phi} &= a(1-\tilde{h}\tilde{f})\sin^2\theta,
\end{aligned} \tag{2}$$

with

$$\Sigma = r^2 + a^2\cos^2\theta, \tag{3}$$

and

$$\begin{aligned}
\tilde{f} &= 1 - \frac{2\tilde{m}_1 r}{\Sigma}, \\
\tilde{h} &= 1 - \frac{\mu\tilde{m}_2 r}{\Sigma^2}.
\end{aligned} \tag{4}$$

In this paper we are interested to investigate two different but approximately similar black holes: rotating modified Hayward and Bardeen black holes where $\tilde{m}_1$ and $\tilde{m}_2$ are different for them which are explained below.

2.1 Rotating Modified Hayward Black Hole

The rotating modified Hayward black hole metric is defined in equation (1) provided $\tilde{m}_1$ and $\tilde{m}_2$ are given by [29, 31]

$$\begin{aligned} \tilde{m}_1 &= M\frac{r^{3+\alpha}\Sigma^{-\frac{\alpha}{2}}}{r^{3+\alpha}\Sigma^{-\frac{\alpha}{2}}+g_1^3 r^{\beta}\Sigma^{-\frac{\beta}{2}}}, \\ \tilde{m}_2 &= M\frac{r^{3+\alpha}\Sigma^{-\frac{\alpha}{2}}}{r^{3+\alpha}\Sigma^{-\frac{\alpha}{2}}+g_2^3 r^{\beta}\Sigma^{-\frac{\beta}{2}}} \end{aligned} \tag{5}$$

Here $g_1^3 = 2Ml^2$ and $g_2^3 = \frac{\mu}{\nu}M$, where μ and ν are positive constants, and l is a parameter with dimensions of length with small scale related to the inverse cosmological constant and also α and β are real numbers. It is easy to check that $\alpha = \beta = a = 0$ yield to non-rotating Hayward black hole and $\mu = 0$ reduced to ordinary Hayward black hole. Horizon structure of rotating modified Hayward black hole given by $g_{rr} = \infty$ from (2) which is exactly similar to the rotating Hayward black hole discussed by the Ref [31].

2.2 Rotating Modified Bardeen Black Hole

The rotating modified Bardeen black hole metric is defined in equation (1) provided $\tilde{m}_1$ and $\tilde{m}_2$ are given by

$$\begin{aligned} \tilde{m}_1 &= M\frac{r^{3+\alpha}\Sigma^{-\frac{\alpha}{2}}}{\left(r^{3+\alpha}\Sigma^{-\frac{\alpha}{2}}+Ml^2 r^{\beta}\Sigma^{-\frac{\beta}{2}}\right)^{\frac{3}{2}}}, \\ \tilde{m}_2 &= M\frac{r^{3+\alpha}\Sigma^{-\frac{\alpha}{2}}}{\left(r^{3+\alpha}\Sigma^{-\frac{\alpha}{2}}+g^2 r^{\beta}\Sigma^{-\frac{\beta}{2}}\right)^{\frac{3}{2}}} \end{aligned} \tag{6}$$

where, as before, l is a parameter with dimensions of length, α and β are real numbers, and g is a constant parameter with mass dimension. Horizons of the black hole given by root of the following equation (denominator of g_{rr} to be zero),

$$\tilde{f}\Sigma + a^2\sin^2\theta = 0. \tag{7}$$

Using the equation (6) we have

$$\Delta = r^2 + a^2 - \frac{2Mr^{4+\alpha}\Sigma^{-\frac{\alpha}{2}}}{\left(r^{3+\alpha}\Sigma^{-\frac{\alpha}{2}}+Ml^2 r^{\beta}\Sigma^{-\frac{\beta}{2}}\right)^{\frac{3}{2}}} = 0. \tag{8}$$

Horizon structure of rotating modified Bardeen black hole given by the plots of Fig. 1. We can see that for the suitable choice of parameters there are two horizons can be written as $r_{\pm} = r \pm \delta$, where $0 < \delta < 0.5$. The case of $\theta = \frac{\pi}{2}$ is our interest, although values of α and β are not important. In this case, Δ vs r is plotted in Figs. 1 (a) and (b). On the other hand for the case of $\theta = \frac{\pi}{6}$ we can see effect of α and β on black hole horizons in Figs. 1 (c) and (d). For example from red solid lines of Fig. 1 (a) and (b) we can see $r_+ \approx 1.1$ and $r_- \approx 0.5$ for $\alpha = 1$, $\beta = 2$ and $a = 0.5$ (Bardeen black hole). Also one can obtain $r_+ \approx 1.65$ and $r_- \approx 0.9$ with $\alpha = 1$, $\beta = 2$ and $a = 0.5$ for the case of Hayward black hole.

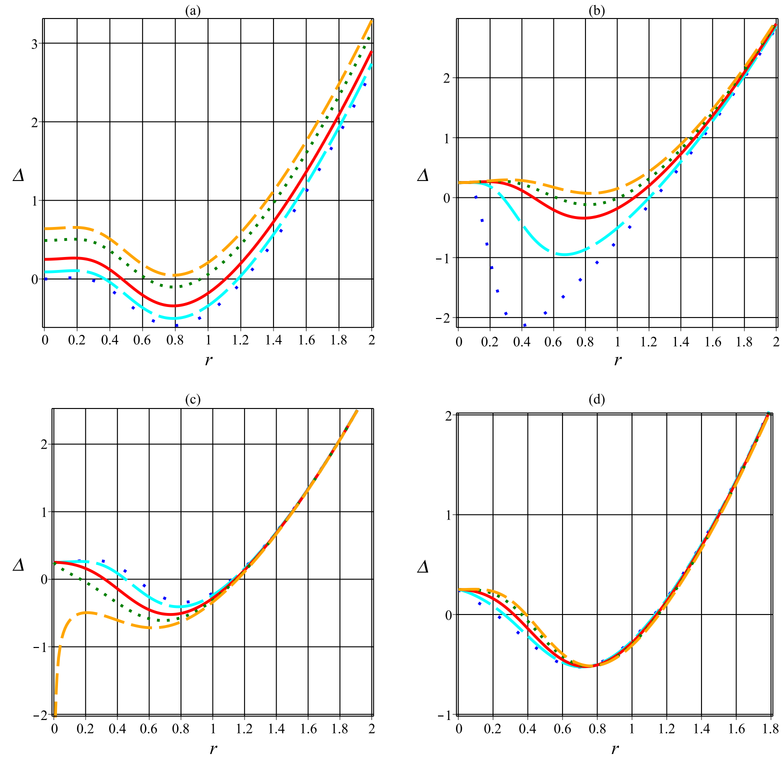

Figure 1: Δ in terms of r for various values of l, a, α and β with $M = 1$, $\theta = \frac{\pi}{2}$ ((a) and (b)), $\theta = \frac{\pi}{6}$ ((c) and (d)). (a) $\alpha = 1$ and $\beta = 2$, $l = 0.5$, $a = 0$ (blue space dot), $a = 0.3$ (cyan long dash), $a = 0.5$ (red solid), $a = 0.7$ (green dot), $a = 0.8$ (orange dash). (b) $\alpha = 1$ and $\beta = 2$, $a = 0.5$, $l = 0.1$ (blue space dot), $l = 0.3$ (cyan long dash), $l = 0.5$ (red solid), $l = 0.6$ (green dot), $l = 0.7$ (orange dash). (c) $a = 0.5$, $l = 0.5$, $\alpha = 1$, $\beta = 0$ (blue space dot), $\beta = 0.8$ (cyan long dash), $\beta = 2$ (red solid), $\beta = 2.8$ (green dot), $\beta = 3.6$ (orange dash). (d) $a = 0.5$, $l = 0.5$, $\beta = 2$, $\alpha = 0$ (blue space dot), $\alpha = 0.4$ (cyan long dash), $\alpha = 1$ (red solid), $\alpha = 1.6$ (green dot), $\alpha = 2$ (orange dash).

3 The Center of Mass Energy

In this section we consider motion of particles with the rest mass m_0 falling from infinity in the background of a rotating modified Hayward or Bardeen black hole. Hamilton-Jacobi equation governs geodesic motion of rotating modified Hayward or Bardeen black hole which can written as,

$$\frac{\partial S}{\partial \tau} = -\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^{\mu}} \frac{\partial S}{\partial x^{\nu}}, \tag{9}$$

where τ is an affine parameter along the geodesics, and S is the Jacobi action, so one can consider the following ansatz [18, 31],

$$S = \frac{1}{2} m_0^2 \tau - Et + L\phi + S_r(r) + S_\theta(\theta), \tag{10}$$

where $S_r(r)$ and $S_\theta(\theta)$ are functions of r and θ respectively. Since equatorial motion ($\theta = \frac{\pi}{2}$) is assumed, so $S_\theta(\theta) = C$ is possible choice, where C is arbitrary constant. Moreover, $E = -P_t$

and $L = P_\phi$ are conserved energy and angular momentum respectively. One can obtain the null geodesic in the following forms:

$$\dot{t} = \frac{1}{\xi}\left[aL(1-\tilde{h}\tilde{f}) - a^2E(\tilde{f}-2) + E\Sigma\right], \tag{11}$$

and

$$\dot{\phi} = \frac{1}{\xi}\left[L\tilde{h}\tilde{f} - a(1-\tilde{h}\tilde{f})E\right] \tag{12}$$

where

$$\xi \equiv a^2\left(\tilde{h}\tilde{f}^2(\tilde{h}-1)+1\right) + \tilde{h}\tilde{f}\Sigma. \tag{13}$$

and *dot* measures the derivative with respect to the parameter τ. Using the Hamilton-Jacobi equation (9) and relation (10), one can obtain,

$$-m_0^2 = g^{tt}E^2 - 2g^{t\phi}EL + g^{\phi\phi}L^2 + g^{rr}R^2(r), \tag{14}$$

where

$$R(r) = \frac{dS_r(r)}{dr}, \tag{15}$$

and

$$\begin{aligned}
g^{tt} &= \frac{a^2(\tilde{f}-2)-\Sigma}{\xi}, \\
g^{rr} &= \frac{a^2+\tilde{f}\Sigma}{\Sigma}, \\
g^{\phi\phi} &= \frac{\tilde{h}\tilde{f}}{\xi}, \\
g^{t\phi} &= \frac{a(1-\tilde{h}\tilde{f})}{\xi},
\end{aligned} \tag{16}$$

It is easy to find,

$$R(r) = \sqrt{\frac{\Sigma}{a^2+\tilde{f}\Sigma}\left[\frac{2ELa(1-\tilde{h}\tilde{f}) - E^2(a^2(\tilde{f}-2)-\Sigma) - L^2\tilde{h}\tilde{f}}{\xi} - m_0^2\right]}. \tag{17}$$

So, we have,

$$\dot{r} = \frac{a^2+\tilde{f}\Sigma}{\Sigma}R(r). \tag{18}$$

Therefore, we have all non-zero 4-velocity components given by equations (11), (12), and (18). Hence we are able to obtain center of mass (CM) energy of two neutral particles collision near the rotating modified Hayward or Bardeen black hole. We suppose that the two particles have the same rest mass (m_0) with the angular momentum per unit mass L_1, L_2 and energy per unit mass E_1, E_2, respectively. In that case CM energy is given by,

$$\epsilon \equiv \tilde{E}_{CM} = \sqrt{1 - g_{\mu\nu}u_1^\mu u_2^\nu}, \tag{19}$$

where $u_i^\mu = (\dot{t}_i, \dot{r}_i, 0, \dot{\phi}_i)$, $i = 1, 2$ (since for equatorial plane $\theta = \pi/2$, so $\dot{\theta} = 0$) and $E_{CM} = \tilde{E}_{CM}\sqrt{2}m_0$. After some calculations we can find,

$$\tilde{E}_{CM}^2 = \frac{1}{\xi^2}\left(\xi^2 + \mathcal{A}L_1L_2 + \mathcal{B}E_1E_2 + \mathcal{C}(E_1L_2 + E_2L_1) - H_1H_2\right), \tag{20}$$

where

$$\begin{aligned}
\mathcal{A} &= a^2\tilde{f}\tilde{h}(\tilde{f}\tilde{h} + \tilde{f}^2\tilde{h} - \tilde{f}^2\tilde{h}^2 - 1) - \tilde{f}^2\tilde{h}^2\Sigma, \\
\mathcal{B} &= \tilde{f}\tilde{h}[a^2(\tilde{f} - 2) - \Sigma]^2 + a^2(1 - \tilde{f}\tilde{h})[a^2(\tilde{f} - 2) - \Sigma], \\
\mathcal{C} &= a(1 - \tilde{f}\tilde{h})\left(a^2(1 - \tilde{f}\tilde{h}) - \tilde{f}\tilde{h}[a^2(\tilde{f} - 2) - \Sigma]\right),
\end{aligned} \tag{21}$$

and

$$H_i = \sqrt{\xi}\sqrt{\xi m_0^2 - 2a(1 - \tilde{f}\tilde{h})E_iL_i + [a^2(\tilde{f} - 2) - \Sigma]E_i^2 + \tilde{f}\tilde{h}L_i^2}, \qquad i = 1, 2. \tag{22}$$

We give numerical analysis of $\tilde{E}_{CM}$ given by the equation (20) for both cases of rotating modified Hayward and Bardeen black holes. From plots of the Fig. 2, we can see that for some cases of rotating modified Hayward, the CM energy has infinite value. On the other hand it has finite constant value. From Figs. 2 (a) and (d) we can see that $E_1 = 0$ or $L_1 = 0$ yields to infinite CM energy at horizon which is expected result. On the other hand, from Figs. 2 (a) and (b) we can see that $E_1 = E_2$ and $L_1 = L_2$ give constant CM energy with finite value. Fig. 2 (c) shows variation of CM energy in terms of rotational parameter a. We can see that, infinitesimal a may give infinite CM energy.

Near horizon limit $r \to r_+$ tells that,

$$\xi|_{r\to r_+} = (\tilde{h} - 1)(\tilde{h}\tilde{f}^2 - a^2). \tag{23}$$

Therefore, CM energy will be infinite if we have $\tilde{h} = 1$ or $\tilde{h}\tilde{f}^2 = a^2$. From green dotted line of Fig. 2 (a) we can see infinite CM energy near inner ($r_- \approx 0.9$) and outer ($r_- \approx 1.65$) horizon. Also, in the case of Bardeen black hole we can see from Fig. 2 (d) and (e) that the CM energy will be infinite.

4 Particles Orbits

In order to specify the range of the particles angular momentum, we should calculate the effective potential for describing the motion of the test particles. In the equatorial plane ($\theta = \pi/2$), the radial equation of motion for the time-like particles moving along the geodesic is described by

$$\frac{1}{2}\dot{r}^2 + V_{eff} = 0 \tag{24}$$

which gives the effective potential,

$$\begin{aligned}
V_{eff} &= -\frac{\dot{r}^2}{2} \\
&= -\frac{a^2 + \tilde{f}\Sigma}{2\Sigma}\left[\frac{2ELa(1 - \tilde{h}\tilde{f}) - E^2(a^2(\tilde{f} - 2) - \Sigma) - L^2\tilde{h}\tilde{f}}{\xi} - m_0^2\right]
\end{aligned} \tag{25}$$

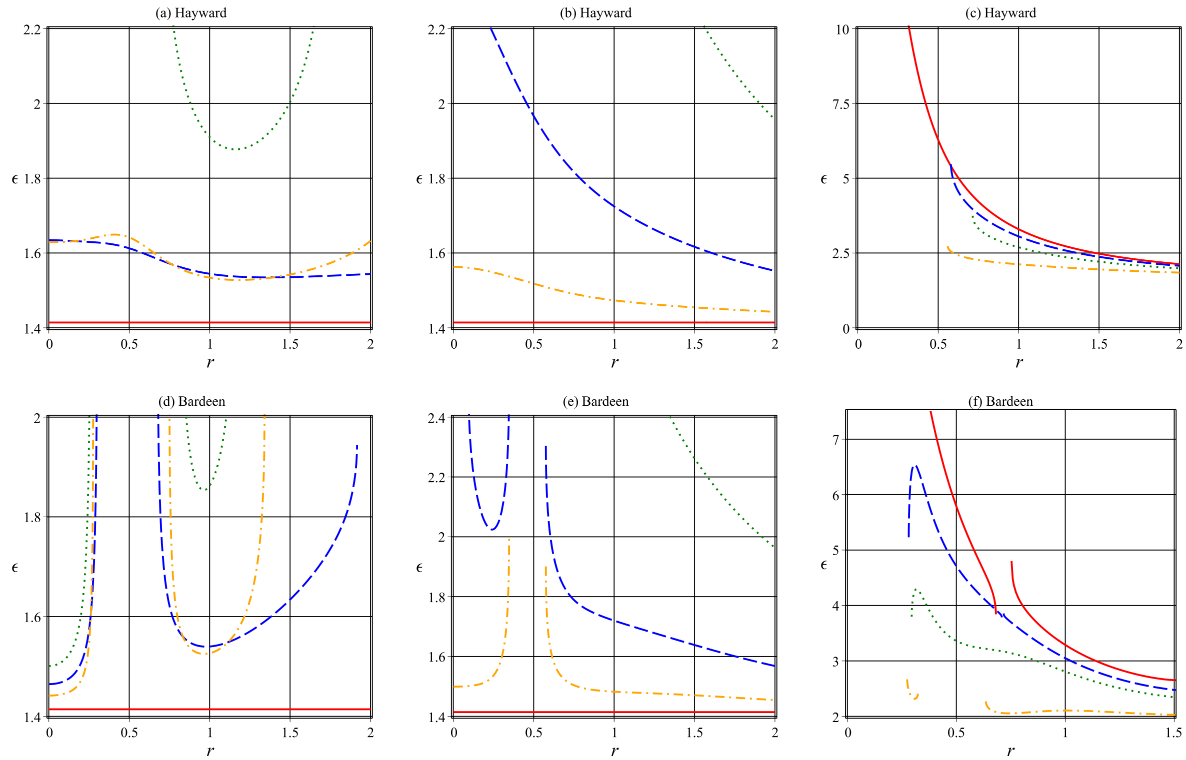

Figure 2: $\epsilon \equiv \tilde{E}_{CM}$ in terms of r for $\alpha = 1$ and $\beta = 2$ with $M = 1$, $l = 0.5$, $\mu = \nu = 1$ and $\theta = \frac{\pi}{2}$. (a) $a = 0.5$, $L_1 = L_2 = 2$. $E_1 = E_2 = 1$ (red solid), $E_1 = 2, E_2 = 1$ (blue dash), $E_1 = 0, E_2 = 1$ (green dot), $E_1 = 0.4, E_2 = 1$ (orange dash dot). (b) $a = 0.5$, $E_1 = E_2 = 2$. $L_1 = L_2 = 2$ (red solid), $L_1 = 0, L_2 = 2$ (blue dash), $L_1 = -2, L_2 = 2$ (green dot), $L_1 = 1, L_2 = 2$ (orange dash dot). (c) $E_1 = 1, E_2 = 2$, $L_1 = 2, L_2 = -2$. $a = 0$ (red solid), $a = 0.2$ (blue dash), $a = 0.4$ (green dot), $a = 1$ (orange dash dot). (d) $g = 0.5$, $a = 0.5$, $L_1 = L_2 = 2$. $E_1 = E_2 = 1$ (red solid), $E_1 = 2, E_2 = 1$ (blue dash), $E_1 = 0, E_2 = 1$ (green dot), $E_1 = 0.4, E_2 = 1$ (orange dash dot). (e) $g = 0.5$, $a = 0.5$, $E_1 = E_2 = 2$. $L_1 = L_2 = 2$ (red solid), $L_1 = 0, L_2 = 2$ (blue dash), $L_1 = -2, L_2 = 2$ (green dot), $L_1 = 1, L_2 = 2$ (orange dash dot). (f) $g = 0.5$, $E_1 = 1, E_2 = 2$, $L_1 = 2, L_2 = -2$. $a = 0$ (red solid), $a = 0.2$ (blue dash), $a = 0.4$ (green dot), $a = 1$ (orange dash dot).

where we have used (17) and (18). The circular orbit of the particles obtained using the following relations,

$$V_{eff} = 0, \tag{26}$$

and

$$W \equiv \frac{dV_{eff}}{dr} = 0. \tag{27}$$

The first condition (26) satisfied at black hole horizon. In the Fig. 3 we can see the behavior of W to satisfied the condition. For the unitary values of a and E we can see that condition (27) satisfied for the modified Hayward black hole with negative L (see Fig. 3 (a)). In the other plots (Fig. 3 (b) and (c)) we can see effect of a and E. On the other hand Fig. 3 (d), (e) and (f) show that rotating modified Bardeen black hole has no restriction for negative L. For any values of L

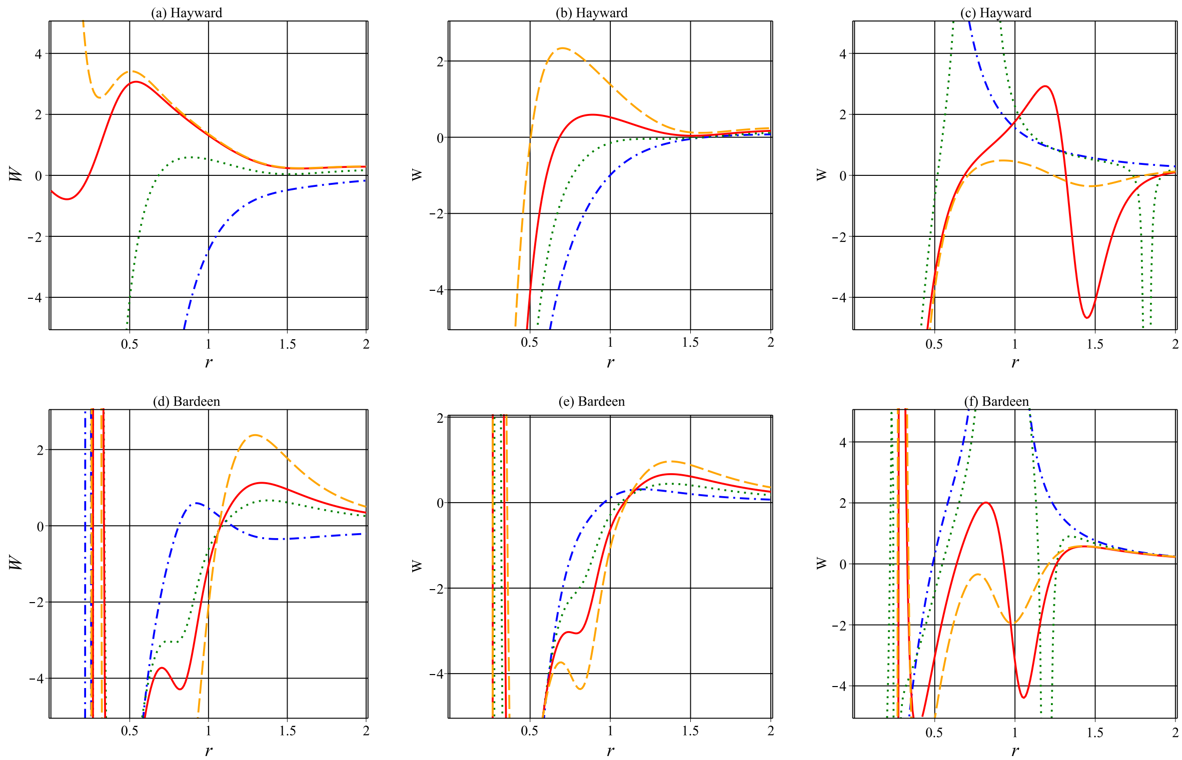

Figure 3: $W \equiv \frac{dV_{eff}}{dr}$ in terms of r for $\alpha = 1$ and $\beta = 2$ with $M = 1$, $l = 0.5$, $g = 0.25$ and $\theta = \frac{\pi}{2}$. Upper plots drawn for Hayward black hole and lower plots drawn for Bardeen black hole. (a) $a = 1$, $E = 1$, $L = -2$ (blue dash dotted), $L = -0.4$ (green dot), $L = 0$ (red solid), $L = 0.02$ (orange dash). (b) $a = 1$, $L = -0.4$, $E = 0$ (blue dash dotted), $E = 0.8$ (green dot), $E = 1$ (red solid), $E = 1.2$ (orange dash). (c) $E = 1$, $L = -0.4$. $a = 0$ (blue dash dotted), $a = 0.2$ (green dot), $a = 0.6$ (red solid), $a = 0.8$ (orange dash). (d) $a = 1$, $E = 1$, $L = -2$ (blue dash dotted), $L = -0.4$ (green dot), $L = 0$ (red solid), $L = 0.8$ (orange dash). (e) $a = 1$, $L = -0.4$, $E = 0$ (blue dash dotted), $E = 0.8$ (green dot), $E = 1$ (red solid), $E = 1.2$ (orange dash). (f) $E = 1$, $L = -0.4$. $a = 0$ (blue dash dotted), $a = 0.2$ (green dot), $a = 0.6$ (red solid), $a = 0.8$ (orange dash).

we have particle circle.

5 Extremal Limit

It may be interesting to study solution at the extremal limit where $\delta = 0$ and $r_+ = r_-$. It will be obtained using appropriate choice of α, β and a. For example, extremal limit of rotating modified Hayward black hole may given by $\alpha = 1$, $\beta = 2$ and $a = 0.65$ (other parameter fixed as previous), in that case $r_+ = r_- \approx 1.25$. Also extremal limit of rotating modified Bardeen black hole may given by $\alpha = 1$, $\beta = 2$, $g = 0.5$ and $a = 1.22$ (other parameter fixed as previous), in that case $r_+ = r_- \approx 0.6$. It is clear from Fig. 4 that infinite CM energy near the black hole horizon will be obtained for both Hayward and Bardeen black holes..

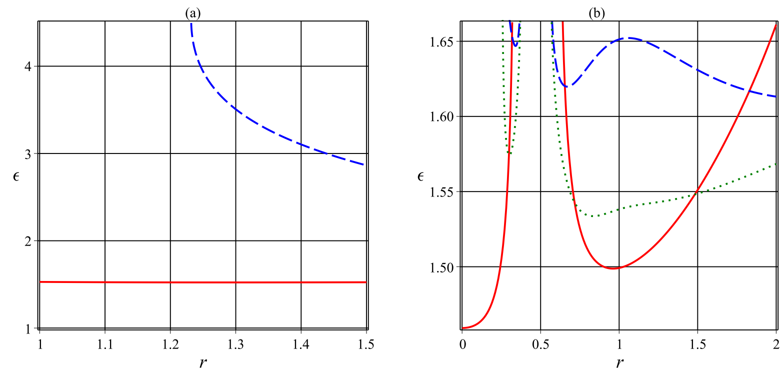

Figure 4: $\epsilon \equiv \tilde{E}_{CM}$ in terms of r for $\alpha = 1$ and $\beta = 2$ with $M = 1$, $l = 0.5$, $\mu = \nu = 1$ and $\theta = \frac{\pi}{2}$. (a) extremal ($a = 0.65$) rotating modified Hayward black hole. $L_1 = L_2 = 2$, $E_1 = 1$, $E_2 = 2$ (red solid), $L_1 = -2$, $L_2 = 2$, $E_1 = 1$, $E_2 = 2$ (blue dash), $L_2 = -2$, $L_1 = 2$, $E_2 = 1$, $E_1 = 2$ (blue dash). (b) extremal ($a = 1.22$) rotating modified Bardeen black hole. $L_1 = L_2 = 2$, $E_1 = 1$, $E_2 = 2$ (red solid), $L_1 = 0$, $L_2 = 2$, $E_1 = 1$, $E_2 = 2$ (blue dash), $L_2 = -2$, $L_1 = 0$, $E_2 = 1$, $E_1 = 2$ (green dot)
.

6 Conclusions

In this work, we have assumed two types of regular black holes i.e., rotating modified Hayward and rotating modified Bardeen black holes as particle accelerators. Horizon structure of rotating modified Hayward black hole given by $g_{rr} = \infty$ which is exactly similar to the rotating Hayward black hole discussed by the Ref [31]. Horizon structure of rotating modified Bardeen black hole given by the plots of Fig. 1. Figs. 1(a), 1(b) show Δ vs r for $\theta = \pi/2$ and Figs. 1(c), 1(d) show Δ vs r for $\theta = \pi/6$. We have investigated the the center of mass (CM) energy of two colliding neutral particles with same rest masses falling from rest at infinity to near the horizons of the mentioned black holes. Figs. 2(a)-(f) show the CM energy vs r for rotating modified Hayward and Bardeen black holes for different cases. We have also investigated the range of the particle's angular momentum and the orbit of the particle. Figs. 3(a)-(c) and Figs. 3(d)-(f) also show W vs r for rotating modified Hayward and Bardeen black holes respectively. We have also studied CM energy corresponding to extremal black holes and obtained infinite CM energy for appropriate black hole parameters and shown in the figures 4(a)-(b).

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References

[1] M. Bañados, J. Silk and S. M. West, Phys. Rev. Lett. 103, 111102 (2009).

[2] K. Lake, Phys. Rev. Lett. 104, 211102 (2010).

[3] K. Lake, Phys. Rev. Lett. 104, 259903 (2010).

[4] T. Harada and M. Kimura, Classical Quantum Gravity 31, 243001 (2014).

[5] S. W. Wei, Y. X. Liu, H. Guo and C.-E. Fu, Phys. Rev. D 82, 103005 (2010).

[6] C. Liu, S. Chen, C. Ding and J. Jing, Phys. Lett. B 701, 285 (2011).

[7] A. Zakria, M. Jamil, JHEP 1505 (2015) 147.

[8] E. Berti, V. Cardoso, L. Gualtieri, F. Pretorius, U. Sperhake, Phys. Rev. Lett. 103, 239001 (2009).

[9] T. Jacobson, T. P. Sotiriou, Phys. Rev. Lett. 104, 021101 (2010).

[10] O. B. Zaslavskii, JETP Lett. 92, 571 (2010).

[11] T. Igata, T. Harada and M. Kimura, Phys. Rev. D 85, 104028 (2012).

[12] M. Banados, B. Hassanain, J. Silk and S. M. West, Phys. Rev. D 83, 023004 (2011).

[13] I. Hussain, Mod. Phys. Lett. A 27, 1250017 (2012).

[14] A. A. Grib, Yu.V. Pavlov, Astropart. Phys. 34, 581 (2011).

[15] T. Harada, M. Kimura, Phys. Rev. D 83, 024002 (2011).

[16] M. Sharif and N. Haider, Astrophys. Space Sci. 346, 111 (2013).

[17] S. W. Wei, Y. X. Liu, H. T. Li and F. W. Chen, JHEP 12, 066 (2010).

[18] S. G. Ghosh, P. Sheoran, M. Amir, Phys. Rev. D 90 (2014) 103006.

[19] J. Sadeghi, B. Pourhassan, Eur. Phys. J. C. 72, 1984 (2012).

[20] J. Sadeghi, B. Pourhassan, H. Farahani, Commun. Theor. Phys. 62, 358 (2014).

[21] M. Patil and P. S. Joshi, Phys. Rev. D 82, 104049 (2010).

[22] M. Patil, P. S. Joshi and D. Malafarina, Phys. Rev. D 83, 064007 (2011).

[23] M. Patil and P. S. Joshi, Classical Quantum Gravity 28, 235012 (2011).

[24] M. Patil, P. S. Joshi, M. Kimura and K. I. Nakao, Phys. Rev. D 86, 084023 (2012).

[25] J. Bardeen, in Proceedings of GR5 (Tiflis, U.S.S.R., 1968).

[26] S. A. Hayward, Phys. Rev. Lett. 96, 031103 (2006).

[27] E. Ayon-Beato, A. Garcya , Phys. Rev. Lett. 80, 5056 (1998).

[28] G. Abbas U. Sabiullah, Astrophys. Space Sci. 352, 769 (2014).

[29] C. Bambi and L. Modesto, Phys. Lett. B 721, 329 (2013).

[30] T. De Lorenzo, C. Pacilioy, C. Rovelli and S. Speziale, arXiv:1412.6015v1 [gr-qc].

[31] M. Amir, S. G. Ghosh, arXiv:1503.08553 [gr-qc].

[32] P. Pradhan, arXiv:1402.2748 [gr-qc].

[33] N. Haider, Open J. Mod. Phys. 1, 1 (2014).

[34] S. G. Ghosh and M. Amir, arXiv: 1506.04382 [gr-qc].